PERMANENT CONFIGURATIONS IN THE PROBLEM OF FIVE BODIES*

BY

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1. **Introduction.** In 1772 Lagrange found that if three bodies of arbitrary masses are placed at the vertices of an equilateral triangle and given the proper initial velocities, they will continue to be at the vertices of an equilateral triangle which may change in size, but not in shape. He also showed that if the three bodies are placed in a straight line and started off suitably, they will remain in a straight line, and the ratios of the mutual distances will be constant. More recently Moulton† has extended the straight line configuration to n bodies, and some particular instances of other configurations have been given by Hoppe,‡ Andoyer,§ and Longley.‖

A permanent configuration is a configuration of n bodies which has the property that the ratio of distances between corresponding bodies is constant. In other words, the figure may change in size, but not in shape. The object of this paper is to determine necessary and sufficient conditions for any plane configuration of five bodies, other than a straight line configuration, to be permanent and to give a detailed analysis of the various types of such configurations, both convex and concave, that can be formed by rearranging the bodies to give pentagons of different shapes. This work may be regarded as an extension of that of MacMillan and Bartky¶ in which the problem of four bodies is treated.

2. **The equations of motion.** Let $m_1, m_2, \cdots, m_5$ with coordinates (x_1, y_1), $(x_2, y_2), \cdots, (x_5, y_5)$, respectively, represent the masses of a plane system of five bodies referred to a set of rectangular axes, with origin at their common center of mass, which rotates with uniform angular velocity ω. If we assume that the bodies attract each other according to the Newtonian law and move in circles around the origin with uniform angular velocity ω, the differential

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equations of motion reduce to

(1)
$$\omega^2 x_i = \sum_{j=1}^{5} \frac{m_j(x_i - x_j)}{r_{i,j}^3},$$
$$\omega^2 y_i = \sum_{j=1}^{5} \frac{m_j(y_i - y_j)}{r_{i,j}^3}, \qquad i = 1, 2, \cdots, 5; j \neq i,$$

where $r_{i,j}$ is the distance between the masses m_i and m_j.

It is to be observed that the equations

(2)
$$\sum_{j=1}^{5} m_i x_i = 0, \qquad \sum_{i=1}^{5} m_i y_i = 0,$$

which express the fact that the origin of coordinates is at the center of mass, follow directly from (1).

Define a constant r_0 by

(3)
$$\omega^2 = \sum_{i=1}^{5} m_i / r_0^3 .$$

The constant r_0 is to replace ω and will be used later in defining the regions for positive masses.

The equations (1), as a consequence of (2) and (3), may now be written in the equivalent form

(4)
$$\sum_{j=1}^{5} S_{i,j}(x_i - x_j)m_j = 0,$$
$$\sum_{j=1}^{5} S_{i,j}(y_i - y_j)m_j = 0, \qquad i = 1, 2, \cdots, 5; j \neq i,$$

where

$$S_{i,j} = S_{j,i} = R_{i,j} - R_0, \quad R_{i,j} = R_{j,i} = 1/r_{i,j}^3, \quad R_0 = 1/r_0^3.$$

If in (4) m_i and m_{i+1}, $(i = 1, 2, \cdots, 5)$, are eliminated, respectively, from the $(i+1)$th and ith equations in x and the corresponding equations in y, and if we denote twice the area of the triangle whose vertices have as coordinates any three of the points (x_i, y_i), $(i = 1, 2, \cdots, 5)$, by D with a pair of numbers (from 1 to 5) as subscripts which do not occur as subscripts in the coordinates of the vertices of the triangle, we have

(5)
$$S_{i,i+3}D_{i+1,i+2}m_i + S_{i+1,i+3}D_{i,i+2}m_{i+1} + S_{i+2,i+3}D_{i,i+1}m_{i+2} = 0,$$
$$S_{i,i+4}D_{i+1,i+2}m_i + S_{i+1,i+4}D_{i,i+2}m_{i+1} + S_{i+2,i+4}D_{i,i+1}m_{i+2} = 0,$$

or

$$m_i/(S_{i+1,i+3}S_{i+2,i+4} - S_{i+1,i+4}S_{i+2,i+3})D_{i,i+2}D_{i,i+1}$$

(6)
$$= m_{i+1}/(S_{i+2,i+3}S_{i,i+4} - S_{i+2,i+4}S_{i,i+3})D_{i+1,i+2}D_{i,i+1}$$
$$= m_{i+2}/(S_{i,i+3}S_{i+1,i+4} - S_{i,i+4}S_{i+1,i+3})D_{i+1,i+2}D_{i,i+2},$$

$$i = 1, 2, \cdots, 5,$$

a system of equations, equivalent to (4), in which the variables x and y do not occur explicitly. It is to be understood here as well as hereafter that the subscripts 6, 7, 8, and 9 are to be replaced by 1, 2, 3, and 4, respectively.

For each value of i in equations (6), two independent relations may be formed, each expressing a mass ratio as a function of the D's and S's. Among these ten mass ratios six distinct equalities exist; and when (6) is substituted in them we obtain, after some reduction,

(7)
$$\frac{S_{i,i+3}S_{i+1,i+2} - S_{i,i+2}S_{i+1,i+3}}{S_{i,i+2}S_{i+1,i+3} - S_{i,i+1}S_{i+2,i+3}} = \lambda_{i+4}, \qquad i = 1, 2, \cdots, 5,$$

where

(8)
$$\lambda_{i+4} = \frac{D_{i+2,i+3}D_{i,i+1}}{D_{i+1,i+2}D_{i,i+3}}.$$

It is to be observed that the subscript on λ corresponds to the missing subscript in the left-hand members of (7).

One may inquire whether the denominators in (6) may not vanish under certain conditions. That this is impossible may be seen by considering the several possible cases.

It should be stated first that in the five-body problem the masses are all different from zero. The form of equations (1) shows that this hypothesis has been made. Suppose, now, that no three of the bodies are in the same straight line (so that no D is zero), and let one of the expressions in S, say $S_{24}S_{35} - S_{25}S_{34}$, vanish. It follows at once from (6) that

(A) $$S_{i,i+1}S_{i+2,i+4} = S_{i,i+2}S_{i+1,i+4} = S_{i,i+4}S_{i+1,i+2}, \qquad i = 1, 2, \cdots, 5.$$

That is, if one denominator in (6) vanishes, all must vanish. But it is impossible to satisfy all of the equations (A) simultaneously, for if it were, then equations (5) would show that a permanent configuration of five bodies is obtained in which the five masses are arbitrary. Let two of these masses approach zero. The result, in so far as finite masses are concerned, is a permanent configuration of three bodies with arbitrary masses. The only such figure is an equilateral triangle. Hence, if the equations (A) are satisfied by a pentagon configuration, any three vertices of this pentagon must form an equilateral triangle. This is geometrically impossible.

Consider next the case where three or more of the bodies are in the same straight line. The case in which they are all in a line has been treated by Moulton, and the method of this paper is not applicable. Outside of Moulton's case there are only two five-body configurations in which some of the denominators in (6) vanish; namely, the square and the rhombus with one mass at each vertex and the fifth mass at the center. These two cases are treated in Examples 2 and 3 on pages 578 and 579, respectively. Hence no omission of cases arises through regarding the denominators in (6) as non-zero.

3. **The relations among the triangular area ratios.** We assume that in general no three of the points (x_i, y_i), $(i = 1, 2, \cdots, 5)$, lie in the same straight line. Hence they uniquely determine a conic, and this conic may be projected into a circle. In this process of projection the triangular area ratios remain unchanged; that is, the λ's defined in (8) are invariant under projection. If we make use of this property, we find that when the D's are eliminated from (8), the λ's satisfy a system of equations of the form

$$(9) \qquad \lambda_i - \lambda_{i+2}\lambda_{i+3} - \lambda_i\lambda_{i+2}\lambda_{i+3} = 0, \qquad\qquad i = 1, 2, \cdots, 5.$$

4. **A geometric property of the λ's.** Let us consider, as typical of the triangular area ratios, the first one, which by (8) is

$$\lambda_5 = D_{12}D_{34}/D_{14}D_{23}.$$

If the points 1, 2, 3, and 4 are assigned coordinates and λ_5 is assigned a value, then this equation represents a conic whose variables are the coordinates of the point 5, the nature of the conic depending upon the value given to λ_5. The conic obviously passes through the points 1, 2, 3, and 4. It will be useful later to know under what condition the conic will degenerate into a pair of straight lines.

Let the points 1, 2, 3, and 4 have, respectively, the coordinates $(0, 0)$, $(x_2, 0)$, (x_3, y_3), and (x_4, y_4), let the coordinates of the point 5 be (x, y), and omit the subscript on λ_5. Then if the equation of the conic is written out and the condition for a conic to degenerate into a pair of straight lines is imposed, it is found that

$$\lambda(\lambda + 1) = 0.$$

Hence we have the following theorem:

THEOREM 4.1. *If, in any one of the triangular area ratios, the point indicated by the subscript on the associated λ is allowed to vary while the other four points are fixed, the variable point will move in a conic which passes through the four fixed points and will degenerate into a pair of straight lines if, and only if, λ has either the value 0 or -1.*

5. The necessary condition. In order that the problem may admit a solution other than the straight line solution, it is necessary that equations (7) be satisfied. However, if any three of these equations are satisfied, the remaining two are also satisfied, since any set of three of them implies the other two. For instance, suppose we write down the following three of these equations:

$$S_{14}S_{23} - S_{13}S_{24}(1 + \lambda_5) + S_{12}S_{34}\lambda_5 = 0$$

(10)
$$S_{15}S_{24} - S_{14}S_{25}(1 + \lambda_3) + S_{12}S_{45}\lambda_3 = 0,$$

$$S_{25}S_{34} - S_{24}S_{35}(1 + \lambda_1) + S_{23}S_{45}\lambda_1 = 0,$$

which correspond, respectively, to the values 1, 4, and 2 of i, and write down the one corresponding to $i = 5$, namely,

(10a) $$S_{12}S_{35} - S_{13}S_{25}(1 + \lambda_4) + S_{15}S_{23}\lambda_4 = 0.$$

To show that this equation is implied by equations (10) multiply the second equation of (10) by $\lambda_1 S_{23}$ and the third one by $\lambda_3 S_{12}$ and subtract, thus eliminating S_{45}. Multiply this result throughout by λ_5 and the first equation of (10) by λ_3. Then, using the relation $\lambda_3 = \lambda_1\lambda_5 + \lambda_1\lambda_3\lambda_5$ from (9) and subtracting, we have

(11) $$S_{13}S_{25}(1 + \lambda_5) - S_{12}S_{35}(1 + \lambda_1)\lambda_5 - S_{15}S_{23}(1 - \lambda_1\lambda_5) = 0.$$

In order to see that this equation is identical with (10a), one only has to eliminate λ_2 from the second and fourth of the λ equations (9) obtaining $\lambda_4 = (1 - \lambda_1\lambda_5)/(1 + \lambda_1)\lambda_5$, and substitute this expression in (10a).

6. Convex and concave pentagons. Suppose that pegs are placed at each of the five masses and a string is drawn tautly around them. Two cases are distinguishable:

Case 1. The string touches all five pegs, and
     (a) no three pegs are in a straight line;
     (b) three pegs are in a straight line, but not four;
     (c) four pegs are in a straight line, but not five;
     (d) five pegs are in a straight line.

Case 2 (a). The string touches only four pegs, the fifth one being inside the quadrilateral formed by the other four.

Case 2 (b). The string touches only three pegs, the other two being inside the triangle formed by the first three.

If the conditions of Case 1 (a), 1 (b), or 1 (c) are satisfied, the pentagon will be called convex. Case 1 (d) is the straight line configuration with which we are not concerned. If the conditions of Case 2 (a) or Case 2 (b) are satisfied, the pentagon will be called concave.

In all cases we shall regard the masses as arranged in the counter-clockwise order m_1, m_2, m_3, m_4, and m_5.

From equations (6) we may write

$$m_1 = \frac{D_{12}}{D_{23}} \frac{(R_{24} - R_0)(R_{35} - R_0) - (R_{34} - R_0)(R_{25} - R_0)}{(R_{14} - R_0)(R_{25} - R_0) - (R_{24} - R_0)(R_{15} - R_0)} m_3,$$

$$m_2 = \frac{D_{12}}{D_{13}} \frac{(R_{34} - R_0)(R_{15} - R_0) - (R_{14} - R_0)(R_{35} - R_0)}{(R_{14} - R_0)(R_{25} - R_0) - (R_{15} - R_0)(R_{24} - R_0)} m_3,$$

$$m_2 = \frac{D_{24}}{D_{34}} \frac{(R_{13} - R_0)(R_{45} - R_0) - (R_{14} - R_0)(R_{35} - R_0)}{(R_{14} - R_0)(R_{25} - R_0) - (R_{12} - R_0)(R_{45} - R_0)} m_3,$$

$$m_4 = \frac{D_{24}}{D_{23}} \frac{(R_{12} - R_0)(R_{35} - R_0) - (R_{13} - R_0)(R_{25} - R_0)}{(R_{14} - R_0)(R_{25} - R_0) - (R_{12} - R_0)(R_{45} - R_0)} m_3,$$

$$m_4 = \frac{D_{45}}{D_{35}} \frac{(R_{13} - R_0)(R_{25} - R_0) - (R_{23} - R_0)(R_{15} - R_0)}{(R_{24} - R_0)(R_{15} - R_0) - (R_{14} - R_0)(R_{25} - R_0)} m_3,$$

(12)

$$m_5 = \frac{D_{45}}{D_{34}} \frac{(R_{14} - R_0)(R_{23} - R_0) - (R_{13} - R_0)(R_{24} - R_0)}{(R_{24} - R_0)(R_{15} - R_0) - (R_{14} - R_0)(R_{25} - R_0)} m_3,$$

$$m_1 = \frac{D_{14}}{D_{45}} \frac{(R_{24} - R_0)(R_{35} - R_0) - (R_{34} - R_0)(R_{25} - R_0)}{(R_{12} - R_0)(R_{34} - R_0) - (R_{13} - R_0)(R_{24} - R_0)} m_5,$$

$$m_4 = \frac{D_{14}}{D_{15}} \frac{(R_{13} - R_0)(R_{25} - R_0) - (R_{12} - R_0)(R_{35} - R_0)}{(R_{12} - R_0)(R_{34} - R_0) - (R_{13} - R_0)(R_{24} - R_0)} m_5,$$

$$m_1 = \frac{D_{12}}{D_{25}} \frac{(R_{23} - R_0)(R_{45} - R_0) - (R_{24} - R_0)(R_{35} - R_0)}{(R_{13} - R_0)(R_{24} - R_0) - (R_{14} - R_0)(R_{23} - R_0)} m_5,$$

$$m_2 = \frac{D_{12}}{D_{15}} \frac{(R_{14} - R_0)(R_{35} - R_0) - (R_{13} - R_0)(R_{45} - R_0)}{(R_{13} - R_0)(R_{24} - R_0) - (R_{14} - R_0)(R_{23} - R_0)} m_5.$$

For a convex pentagon, the ten triangular area ratios in these equations satisfy the inequalities

(13) $D_{12}/D_{23} > 0$, $D_{12}/D_{13} > 0$, $D_{24}/D_{34} > 0$, $D_{24}/D_{23} > 0$,
$D_{45}/D_{35} > 0$, $D_{45}/D_{34} > 0$, $D_{14}/D_{45} > 0$, $D_{14}/D_{15} > 0$,
$D_{12}/D_{25} > 0$, $D_{12}/D_{15} > 0$.

For the general concave pentagon of the type described in Case 2 (a), there are four different positions that the fifth mass may occupy within the quadrilateral formed by the other four masses. These positions are shown in Figs. 1a, 1b, 1c, and 1d. The following table gives the signs of the triangular area ratios for each of these four cases:

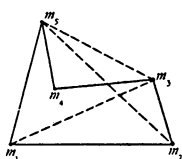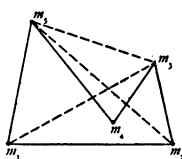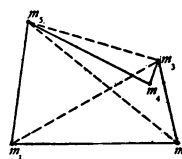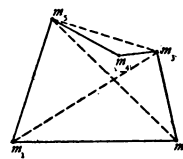| Figure	$\dfrac{D_{12}}{D_{23}}$	$\dfrac{D_{12}}{D_{13}}$	$\dfrac{D_{24}}{D_{34}}$	$\dfrac{D_{24}}{D_{23}}$	$\dfrac{D_{45}}{D_{35}}$	$\dfrac{D_{45}}{D_{34}}$	$\dfrac{D_{14}}{D_{45}}$	$\dfrac{D_{14}}{D_{15}}$	$\dfrac{D_{12}}{D_{25}}$	$\dfrac{D_{12}}{D_{15}}$
1a	−	+	+	+	+	+	+	+	−	−
1b	−	+	+	+	+	+	+	+	+	−
1c	−	−	+	+	+	+	+	+	+	−
1d	−	−	+	+	+	+	+	+	−	−

FIG. 1a   FIG. 1b   FIG. 1c   FIG. 1d

7. Admissible convex pentagons. An admissible pentagon is one for which there exist positive masses which, when placed at its vertices and started off suitably, will give a permanent configuration.

In any solution of the problem, one sees from (10) that the following equations

$$
(14)\quad
\begin{aligned}
\frac{(R_{14} - R_0)(R_{23} - R_0) - (R_{13} - R_0)(R_{24} - R_0)}{(R_{13} - R_0)(R_{24} - R_0) - (R_{12} - R_0)(R_{34} - R_0)} &= \frac{D_{12}D_{34}}{D_{14}D_{23}}, \\[2mm]
\frac{(R_{24} - R_0)(R_{15} - R_0) - (R_{14} - R_0)(R_{25} - R_0)}{(R_{14} - R_0)(R_{25} - R_0) - (R_{12} - R_0)(R_{45} - R_0)} &= \frac{D_{12}D_{45}}{D_{24}D_{15}}, \\[2mm]
\frac{(R_{34} - R_0)(R_{25} - R_0) - (R_{24} - R_0)(R_{35} - R_0)}{(R_{24} - R_0)(R_{35} - R_0) - (R_{23} - R_0)(R_{45} - R_0)} &= \frac{D_{23}D_{45}}{D_{34}D_{25}}
\end{aligned}
$$

must be satisfied, and in addition, for any solution, equations (12) must yield positive masses.

Since for convex pentagons the inequalities (13) must hold, and since the masses are necessarily positive, it follows that the numerator and denominator involving the R's in each of the equations (12) must be of like sign. If we let the numerator and denominator of the first equation be positive, the signs of the numerator and denominator in each of the other equations are determined, and we find that

$$(15) \quad (R_{i,i+1} - R_0)(R_{i+2,i+3} - R_0) > (R_{i,i+2} - R_0)(R_{i+1,i+3} - R_0)$$
$$> (R_{i,i+3} - R_0)(R_{i+1,i+2} - R_0), \quad i = 1, \cdots, 5.$$

We can choose three of the points arbitrarily. Let these be the first three. We then have given r_{12}, r_{23}, and r_{13} and can therefore choose their order relation. Let this relation be

$$r_{12}, r_{23} \leqq r_0 \leqq r_{13}.$$

Draw r_{12} as in Fig. 2, and let m_1 and m_2 be at its extremities. With m_1 and m_2 as centers draw semicircles with radius r_0, intersecting at O. With O as center describe circular arc $a_1b_1c_4b_2a_2$ with radius r_0. Then with m_3 in the region Oa_2b_2, m_4 in the region bounded by circular arcs Ob_1, Ob_2, and $b_1c_4b_2$, and m_5 in the region Oa_1b_1, the inequalities (15) are all satisfied, subject to the condition

$$(16) \quad r_{12}, r_{23}, r_{34}, r_{15}, r_{45} \leqq r_0 \leqq r_{13}, r_{14}, r_{24}, r_{25}, r_{35}$$

which assures that the masses in the regions specified are all positive. The proof that with the masses in these regions equations (14) are satisfied is too long to write down here. Only the method of procedure will be indicated.

In Fig. 2 take m_3 in its region as shown or on the boundary of this region, and with this point as center and with r_0 as radius describe circular arc $c_1c_2c_3c_4$, intersecting arc Oa_1 at c_2, arc Ob_1 at c_3, and arc b_1b_2 at c_4. On account of the inequalities (16), the mass m_4 is further restricted to the region $Oc_3c_4b_2$. Let this region be covered with an infinity of arcs, all passing through the point O and intersecting the boundary arc c_3c_4. It can be shown that on each of these arcs there is one and but one point at which the first of equations (14) is satisfied. If these points are joined by a curve, we have the curve on which point m_4 moves as m_3 varies its position in the region Oa_2b_2. Imagine m_4 on this curve, and with it as center and r_0 as radius describe circular arc a_3b_3 intersecting Ob_1 at b_3 and Oa_1 at some point a_3 below c_2. The remaining two equations of (14), then, determine two curves which pass through m_4 and intersect in the region $c_2c_3b_3a_3$, giving the position of m_5. As the point m_4 varies its position on its curve, m_5 moves on some curve in the region $c_2c_3b_3a_3$. These results lead to the following theorem:

THEOREM 7.1. *For every point m_3 in the region Ob_2a_2, or on the boundary of this region, there exists a single infinity of points m_4 in the region $Oc_3c_4b_2$ and a single infinity of points m_5 in the region Oa_1b_1, which together with the points m_1 and m_2 form an admissible convex pentagon.*

8. **Limitations on the interior angles.** In Fig. 2 let the angles of the pentagon at its five vertices be denoted by the masses at these points. For example, angle $m_2m_1m_5$ will be denoted by m_1, and angle $m_1m_2m_3$ by m_2.

It is obvious that angle m_1 is equal to or greater than angle m_2m_1O and that this latter angle obtains its least value when $r_{12}=r_0$. This value is seen to be 60° from Fig. 3, which is the same as Fig. 2 except that it is drawn with $r_0=r_{12}$. Therefore angle m_1 is equal to or greater than 60°, and the same is clearly true for angle m_2.

Since $r_{12}=r_0$, the first of equations (14) shows that the point m_4 moves on some curve through the point O, such as curve n, Fig. 3. We see, however, that the smallest value of angle m_3 is obtained when point m_3 is at b_2 and point m_4 is at O, in which case it is 60°. By symmetry it follows that the smallest value of angle m_5 is also 60° and is obtained when point m_4 is at O and point m_5 is at b_1.
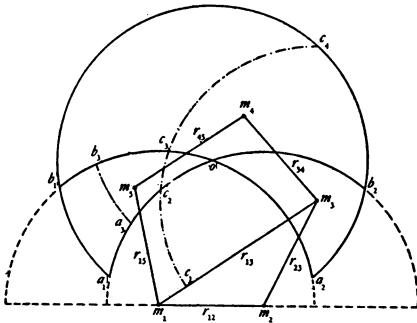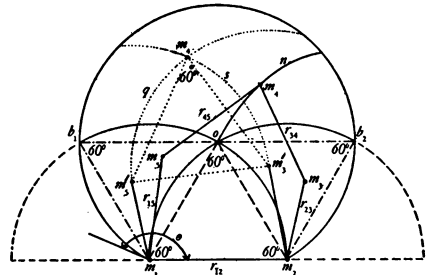

FIG. 2   FIG. 3

Consider now angle m_4. In Fig. 3, let points m_3 and m_5 be at m_3' and m_5', respectively, so that r_{35} equals r_0, its least value. With these two points as centers describe arcs q and s, respectively, each with radius $r_0=r_{12}=r_{35}$. From the inequality (16) we see that the point m_4 cannot lie outside of the region bounded by the arcs q, s and the arcs of the circles drawn with the points m_1 and m_2 as centers. It is clear then that angle m_4 is smallest when point m_4 lies at the intersection of arcs q and s, and that it is then 60°.

Hence no interior angle of the convex pentagon can be less than 60°.

The maximum value of the angle m_1 is also reached when $r_{12}=r_0$, as can be seen from Figs. 2 and 3, and is that angle θ, Fig. 3, which the circle with O as center makes with the line r_{12}.

In order to find θ let the line through the points m_1 and m_2 be the x axis, and let these two points be symmetric with respect to the origin, so that the point O will be on the y axis. Then if the equation of the circle with O as center and r_0 as radius is formed, it follows easily that the angle θ which it makes with the line r_{12} is 150°.

It is obvious from Fig. 3 that angle m_2 also has this angle for its maximum value; and this is true for angles m_3, m_4, and m_5 as well, as can be shown by going through the same procedure for these angles as for angle m_1.

THEOREM 8.1. *Each of the interior angles of an admissible convex pentagon lies between 60° and 150°.*

9. **Admissible concave pentagons.** We shall consider the general case where one of the points m_i is within the quadrilateral formed by the other four. Let the vertices of the quadrilateral be m_1, m_2, m_3, and m_5, and let the interior point be m_4 whose position is that shown in Fig. 1d.

For concave pentagons of the type under consideration, the signs indicated in the last row in the table on page 569 must obtain, and since the masses are positive, the R's in equations (12) must satisfy the relations

$$(R_{12} - R_0)(R_{45} - R_0) > (R_{14} - R_0)(R_{25} - R_0) < (R_{24} - R_0)(R_{15} - R_0),$$

$$(R_{34} - R_0)(R_{15} - R_0) > (R_{14} - R_0)(R_{35} - R_0) > (R_{13} - R_0)(R_{45} - R_0),$$

$$(17) \quad (R_{34} - R_0)(R_{25} - R_0) < (R_{24} - R_0)(R_{35} - R_0) > (R_{23} - R_0)(R_{45} - R_0),$$

$$(R_{23} - R_0)(R_{15} - R_0) < (R_{13} - R_0)(R_{25} - R_0) > (R_{12} - R_0)(R_{35} - R_0),$$

$$(R_{14} - R_0)(R_{23} - R_0) > (R_{13} - R_0)(R_{24} - R_0) < (R_{12} - R_0)(R_{34} - R_0).$$

If m_4 is in the position shown in either Fig. 1a or Fig. 1b, then the first inequality sign in the second inequality and the second inequality sign in the third inequality must be reversed, while there is no change if m_4 is in the position shown in Fig. 1c.

Here, as in the convex case, we may select three of the five points of the pentagon arbitrarily, and we shall take as these three the points m_1, m_2, and m_3. Also let the order relation between r_{12}, r_{23}, and r_{13} be r_{12}, $r_{23} \leqq r_0 \leqq r_{13}$.

In Fig. 4 draw r_{12} with the masses m_1 and m_2 at its extremities. With these two points as centers draw semicircles with radius r_0 intersecting at O. With O as center describe circular arcs a_1b_1 and a_2b_2, each with radius r_0. On account of the inequalities (17), m_3 will lie in the region Ob_2a_2, m_4 in Od_1d_2, and m_5 in Oa_1b_1, with the additional condition

$$r_{12}, r_{23}, r_{34}, r_{14}, r_{24}, r_{15}, r_{25}, r_{45} \leqq r_0 \leqq r_{13}, r_{25}.$$

With the masses in these regions it may be shown that equations (14) are satisfied. Hence we have the companion theorem:

THEOREM 9.1. *For every point m_3 in the region Ob_2a_2, or on the boundary of this region, there is a single infinity of points m_4 in the region Od_1d_2 and a single infinity of points m_5 in the region Oa_1b_1 which, together with the points m_1 and m_2, form an admissible concave pentagon.*

10. Symmetric pentagons. Suppose that

$$r_{23} = r_{15}, \qquad r_{14} = r_{24}, \qquad r_{34} = r_{45}, \qquad r_{13} = r_{25},$$

and consequently that

$$D_{23} = D_{15}, \qquad D_{45} = D_{34}, \qquad D_{14} = D_{24}, \qquad D_{25} = D_{13}.$$

The configurations Fig. 5 then present themselves in the form of isosceles trapezoids with vertices at the points m_1, m_2, m_3, and m_5 and with the point m_4 somewhere on the line perpendicular to and bisecting the parallel sides r_{12} and r_{35}. The construction of the figure is the same as that of Fig. 2.
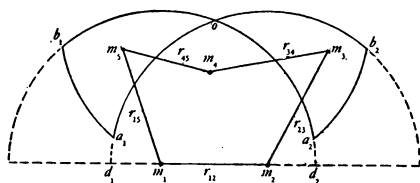

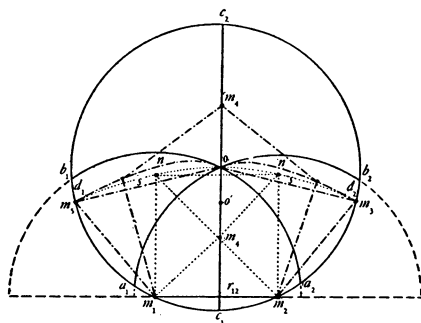FIG. 4   FIG. 5

Equations (14) reduce to the two independent equations

$$(18.1) \qquad \frac{(R_{24} - R_0)(R_{23} - R_{13})}{(R_{13} - R_0)(R_{24} - R_0) - (R_{12} - R_0)(R_{34} - R_0)} = \frac{D_{12}D_{45}}{D_{24}D_{15}},$$

$$(18.2) \qquad \frac{(R_{13} - R_0)(R_{34} - R_0) - (R_{24} - R_0)(R_{35} - R_0)}{(R_{24} - R_0)(R_{35} - R_0) - (R_{23} - R_0)(R_{34} - R_0)} = \frac{D_{15}}{D_{25}},$$

and the mass equations (12) become

$$(19.1) \qquad \frac{m_3}{m_1} = \frac{m_5}{m_2} = \frac{D_{15}}{D_{12}} \frac{(R_{24} - R_0)(R_{13} - R_{23})}{(R_{24} - R_0)(R_{35} - R_0) - (R_{13} - R_0)(R_{34} - R_0)},$$

$$(19.2) \qquad \frac{m_3}{m_2} = \frac{m_5}{m_1} = \frac{D_{25}}{D_{12}} \frac{(R_{24} - R_0)(R_{13} - R_{23})}{(R_{23} - R_0)(R_{34} - R_0) - (R_{24} - R_0)(R_{35} - R_0)},$$

$$(19.3) \qquad \frac{m_3}{m_4} = \frac{m_5}{m_4} = \frac{D_{15}}{D_{24}} \frac{(R_{12} - R_0)(R_{34} - R_0) - (R_{13} - R_0)(R_{24} - R_0)}{(R_{13} - R_0)(R_{13} - R_0) - (R_{12} - R_0)(R_{35} - R_0)},$$

$$m_5 = m_3.$$

From either (19.1) or (19.2) it follows that we have also $m_1 = m_2$.

For convex pentagons all of the triangular area ratios in these equations are positive. For concave pentagons D_{15}/D_{12} and D_{25}/D_{12} are negative, if the point m_4 is above the diagonal r_{13}; otherwise they are positive. Moreover $D_{15}/D_{24}>0$.

The convex case. In Fig. 5, r_{12} is regarded as fixed with the masses m_1 and m_2 at its extremities, while the point m_3 (and its symmetric point m_5) varies when the point m_4 moves from O towards c_2, as m_4 must do since $r_{24} \geqq r_0$ for convex pentagons.

When the point m_4 is at O, equations (18.1) and (18.2) become, respectively,

$$(R_{12} - R_0)(R_{34} - R_0)D_{12}D_{45} = 0,$$

$$(R_{34} - R_0)[(R_{13} - R_0)D_{25} + (R_{23} - R_0)D_{15}] = 0.$$

In the first equation since neither $(R_{12}-R_0)$, D_{12}, nor D_{45} is zero, we must have $(R_{34}-R_0)=0$; whence $r_{34}=r_0$. It then follows that point m_3 is somewhere on the circular arc $m_1b_1c_2b_2a_2m_2$, and it can be proved that it is at some point d_2 between a_2 and b_2 with m_5 at the symmetric position d_1.

As the point m_4 moves from O towards c_2, point m_3 moves on some curve n from d_2 towards O, and in like manner, point m_5 moves from d_1 towards O along some curve n' which is symmetric to n. When m_4 reaches a position on Oc_2 such that the points m_3 and m_5 have moved in towards each other on curves n and n', respectively, so that $r_{35}=r_0$, then the limit for permanent convex configurations is attained, since by inequalities (16), $r_{35} \geqq r_0$.

The concave case. As the point m_4 moves from O towards c_1, as it must do for concave pentagons since $r_{24} \leqq r_0$, point m_3 moves on some curve s from d_2 towards O, and in like manner, point m_5 moves on some curve s', symmetric to s from d_1 towards O. However, since for concave pentagons $r_{35} \leqq r_0$, these curves are of no interest until points m_3 and m_5 reach positions on curves s and s', respectively, where $r_{35}=r_0$.

Let the corresponding position of m_4 be O'. If one starts with the point m_4 in this position, as it moves on towards c_1 points m_3 and m_5 move towards O. When they reach this position one has $r_{13}=r_{23}=r_0$, $D_{12}=0$, $D_{15}/D_{25}=-1$. Equations (18.1) and (18.2) are then both satisfied, regardless of the position of point m_4 on Oc_1.

Thus, for a given r_{12} and r_0, as point m_4 moves from O towards c_2, point m_3 moves from d_2 towards O along n, and point m_5 moves from d_1 towards O along n', giving, in all positions for which $r_{35} \geqq r_0$, a convex pentagon; while as point m_4 moves from O' towards c_1, points m_3 and m_5 move towards O along s and s', respectively, giving, in all positions where $r_{35} \leqq r_0$, a concave pentagon. Since the mass ratios are all positive for all of these pentagons, as

can be seen from equations (19.1), (19.2), and (19.3), we can state the follow-
ing theorem:

THEOREM 10.1. *For every r_{12} and r_0 there exists a single infinity of perma-
nent configurations with one mass at each of the vertices of an isosceles trapezoid
and with the fifth on the line perpendicular to and bisecting the two parallel sides.*

As the point m_4 moves from O towards c_2, Fig. 5, and consequently, points
m_3 and m_5 move from d_2 and d_1 towards the symmetric points on n and n'
where $(r_{35} - r_0)$ vanishes, one sees from equations (19.1), (19.2), and (19.3)
that the mass ratios

$$\frac{m_3}{m_1} = \frac{m_5}{m_2}, \qquad \frac{m_3}{m_2} = \frac{m_5}{m_1}, \qquad \frac{m_3}{m_4} = \frac{m_5}{m_4},$$

change steadily from the values

$$\frac{D_{15}(R_{13} - R_{23})}{D_{12}(R_{35} + R_{13} - 2R_0)}, \qquad \frac{D_{25}(R_{13} - R_{23})}{D_{12}(2R_0 - R_{23} - R_{35})}, \qquad 0,$$

respectively, to

$$\frac{D_{15}(R_{24} - R_0)(R_{13} - R_{23})}{D_{12}(R_0 - R_{13})(R_{34} - R_0)}, \qquad \frac{D_{25}(R_{24} - R_0)(R_{13} - R_{23})}{D_{12}(R_{23} - R_0)(R_{34} - R_0)},$$

$$\frac{D_{15}(R_{12} - R_0)(R_{34} - R_0) - (R_{13} - R_0)(R_{24} - R_0)}{D_{24}(R_{13} - R_0)^2},$$

where the D's and R's are to be evaluated at the points in question. Similar
limits on the mass ratio could be found for the concave case as m_4 moves
from O' towards c_1. We therefore have the following theorem:

THEOREM 10.2. *For every $m_1 = m_2 > 0$, $m_3 = m_5 > 0$, and $m_4 > 0$, there is one
and only one isosceles trapezoid configuration.*

11. **The case of three points on a line.** Let m_4 be on the line r_{13} joining m_1
and m_3, and let m_2 and m_5 be equidistant from r_{13}. We have at once

(20) $\quad$ $r_{12} = r_{15}, \qquad r_{23} = r_{35}, \qquad r_{24} = r_{45},$

$\qquad\quad$ $D_{25} = 0, \qquad D_{35} = D_{23}, \qquad D_{45} = D_{24}, \qquad D_{15} = -D_{12},$

and the equations (14) reduce to the single equation

(21) $\quad$ $\dfrac{(R_{14} - R_0)(R_{23} - R_0) - (R_{13} - R_0)(R_{24} - R_0)}{(R_{13} - R_0)(R_{24} - R_0) - (R_{12} - R_0)(R_{34} - R_0)} = \dfrac{-D_{34}D_{15}}{D_{14}D_{35}}.$

Let points m_1 and m_3 be fixed. Then for a given m_2 and for any position of
point m_4, the equation (21) to be satisfied, by the coordinates of point m_5,

is the equation of a straight line. But for a solution the coordinates of m_5 must also satisfy the first two equations of (20). (The remaining equations of (20) are then automatically satisfied.) These equations, when rationalized, represent circles with centers at the points m_1 and m_3 and with radii r_{12} and r_{23}, respectively.

In Fig. 6 draw r_{13} with m_1 and m_3 at its extremities. Take point m_2 anywhere not on this line, and draw the circles of the above equations. It is obvious that these two equations are satisfied by real points if, and only if, point m_5 lies at the intersection of their circles. That is, point m_5 will either be at the point a, Fig. 6, or it will coincide with point m_2. Thus it may be in one of two positions in so far as these two equations are concerned.

Equation (21) must be satisfied also. By Theorem 4.1, the left member of this equation must be either 0 or -1. If it is 0, the equation reduces to

$$D_{34} = 0,$$

which is the equation of a straight line passing through points m_1 and m_2; therefore, in order for the point m_5 to satisfy all the equations, it must coincide with m_2. This is a case of no interest as r_{25} may vanish for neither convex nor concave pentagons.

Suppose the left-hand member has the value -1. This gives the equation

(22)
$$\frac{R_{12} - R_0}{R_{23} - R_0} = \frac{R_{14} - R_0}{R_{34} - R_0}.$$

The question arises as to whether or not the line with this equation can be made to pass through the point a. In other words, are there configurations other than the one where points m_2 and m_5 coincide? The proof that the answer to this question is in the affirmative is not difficult, but an illustrative example would, perhaps, be more instructive.

Let the point m_4 be midway between points m_1 and m_3. The equation (22) is always satisfied, and if the point m_2 is given a position such that the mass ratios are all positive, we have a permanent configuration which, in general, will be a rhombus with vertices at m_1, m_2, m_3, and m_5, and with m_4 at its center.

In Fig. 7, draw r_{13}, and consider the point m_2 as it moves on a line perpendicular to r_{13} at its mid-point m_4. The positions that it may assume are limited only by the regions for positive masses, since the condition (22) is satisfied for any position of m_2 regardless of what r_0 may be. Moreover wherever point m_2 may be on this line, we have, from (12),

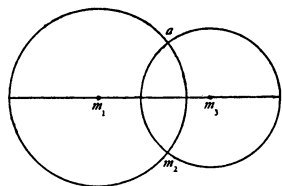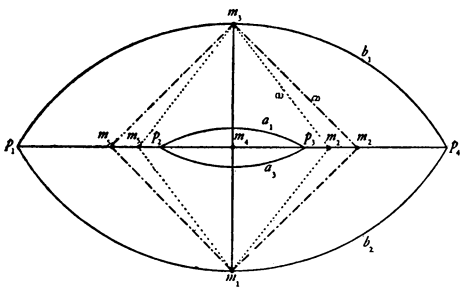$$\frac{m_1}{m_3} = 1, \qquad \frac{m_2}{m_5} = 1.$$

FIG. 6

FIG. 7

With points m_1 and m_3 as centers draw circular arcs a_1 and a_3 with a radius r_0, $(r_0 < r_{13})$, such that the distance between their intersections p_2 and p_3 is equal to r_0. Since $r_{25} \geqq r_0$, we have, with point m_2 at p_3 and m_5 at p_2, the rhombus configuration in which the point m_2 is as close to m_5 as possible for a given r_{13}. In this case the mass equations (12) show that

$$(23) \quad \frac{m_3}{m_2} = 0, \quad \frac{m_4}{m_3} = \frac{2(R_0 - R_{13})}{(R_{14} - R_0)} > 0, \quad \frac{m_3}{m_5} = 0, \quad \frac{m_1}{m_5} = 0, \quad \frac{m_4}{m_5} = 0.$$

The greatest possible distance between the points m_2 and m_5, for a given r_{13}, is obtained when we choose an $r_0 = r_{13}$. Draw circular arcs b_1 and b_2 with such a radius. Then, with m_2 and p_4 and m_5 at p_1, we have the rhombus configuration in which the distance between points m_2 and m_5 is as great as possible. Here, we have

$$(24) \quad \frac{m_3}{m_2} = \infty, \quad \frac{m_4}{m_3} = 0, \quad \frac{m_3}{m_5} = \infty, \quad \frac{m_1}{m_5} = \infty, \quad \frac{m_4}{m_5} = \frac{2(R_0 - R_{25})}{(R_{24} - R_0)} > 0.$$

THEOREM 11.1. *Given r_{13} with masses m_1 and m_3 at its extremities and with the mass m_4 at its center, for a properly chosen r_0, and for all positions of m_2 and m_5 on a line perpendicular to r_{13} at m_4 such that the mass ratios have values lying between those given in (23) and those in (24), the configuration is permanent and is a rhombus with vertices at m_1, m_2, m_3, and m_5, and with m_4 at its center.*

In concluding this section we may add that if any four of the masses are on the same straight line, the fifth mass will be on this line also. This follows readily from equations (14).

12. **The triangle with two interior masses.** One other possibility of a concave pentagon is that of a triangle with one mass at each vertex and with the two remaining masses in its interior. That no such configuration, however, can form an admissible pentagon follows from the inequalities

$$r_{12}, r_{23}, r_{34}, r_{14}, r_{24}, r_{15}, r_{35}, r_{45} \leqq r_0 \leqq r_{13}, r_{25}$$

(see §9). To construct a pentagon of this type one must alter the above order relation that must exist among the sides and diagonals, and the result is always to cause some of the masses to be negative.

13. **Numerical solutions.** We consider the following examples:

EXAMPLE 1. An interesting example of a convex permanent configuration is that of the regular pentagon. In Fig. 5 as the point m_4 moves upward from O towards c_2 and the points m_3 and m_5 move towards each other along curves n and n', somewhere along these paths the configuration becomes that of a regular pentagon. When this position is reached, the equations (18.1) and (18.2) reduce to the single equation

$$(25) \qquad \frac{R_{24} - R_0}{2R_0 - R_{12} - R_{24}} = \frac{D_{15}}{D_{25}} = \frac{r_{12}}{r_{24}},$$

or

$$(26) \qquad (r_0^3 - r_{24}^3)r_{12}^2 r_{24} - 2r_{12}^3 r_{24}^3 + r_0^3(r_{24}^3 + r_{12}^3) = 0.$$

For any given value of r_{13}, r_{24} may be computed from the obvious relation

$$r_{24} = 1.6180 r_{12},$$

and r_0 may be found from (26). It follows readily that

$$\frac{r_0}{r_{12}} = 1.3076, \qquad \frac{r_{23}}{r_{12}} = \frac{r_{34}}{r_{12}} = \frac{r_{45}}{r_{12}} = \frac{r_{15}}{r_{12}} = 1,$$

$$\frac{r_{13}}{r_{12}} = \frac{r_{24}}{r_{12}} = \frac{r_{14}}{r_{12}} = \frac{r_{25}}{r_{12}} = \frac{r_{35}}{r_{12}} = 1.6180,$$

and from (19.1), (19.2), (19.3), together with (25), that $m_1 = m_2 = m_3 = m_4 = m_5$.

EXAMPLE 2. Perhaps the most simple concave permanent configuration in the five-body problem is that of a square with one mass at each of its vertices and with the fifth mass at its center. In Fig. 5 this particular configuration is passed through as the point m_4 moves from O towards c_1 and as points m_3 and m_5 move towards each other along curves s and s'. Equations (18.1) and (18.2) are now identically satisfied for any value of r_0, while the mass equation (19.1) shows that

$$\frac{m_3}{m_1} = \frac{m_5}{m_2} = 1;$$

and since for all of the symmetric configurations, Fig. 5, $m_3 = m_5$, $m_1 = m_2$, we have $m_1 = m_2 = m_3 = m_5$.

The equation (19.3) becomes

$$(27) \quad \frac{m_3}{m_4} = \frac{m_5}{m_4} = \frac{1}{2} \frac{R_{34} - R_0}{2R_0 - R_{12} - R_{13}} = \frac{(r_0^3 - r_{34}^3)r_{12}^3 r_{13}^3}{2r_{34}^3(2r_{12}^3 r_{13}^3 - r_0^3 r_{13}^3 - r_0^3 r_{12}^3)}$$

$$= \frac{(8r_0^3 - 2(2)^{1/2} r_{12}^3)}{2[4(2)^{1/2}r_{12}^3 - r_0^3(1 + 2(2)^{1/2})]},$$

since $r_{13}^2 = 2r_{12}^2 = 4r_{34}^2$.

Thus, while equations (18.1) and (18.2) are satisfied for any r_0, the mass ratios $m_3/m_4 = m_5/m_4$ are not necessarily positive for any value of r_0. It is found that for the last right-hand member of (27) to be positive we must have

$$0.878 < \frac{r_{12}}{r_0} \leqq 1.000.$$

It then follows that

$$0.439 < \frac{m_3}{m_4} = \frac{m_5}{m_4} \leqq 1.414.$$

This result shows that the central mass is arbitrary within certain limits. In general, if a permanent configuration is given, the masses are uniquely determined.

EXAMPLE 3. In Fig. 7 consider the rhombus configuration (1) with masses m_1, m_2, m_3, and m_5 at its vertices and with the mass m_4 at its center. By Theorem 11.1, for a given r_0 the point m_2 may be anywhere on the line perpendicular to r_{13} at its mid-point such that the mass ratios are positive. It is readily found that

$$\frac{r_0}{r_{12}} = 1.042, \qquad \frac{r_{23}}{r_{12}} = \frac{r_{35}}{r_{12}} = \frac{r_{15}}{r_{12}} = 1, \qquad \frac{r_{14}}{r_{12}} = \frac{r_{34}}{r_{12}} = 0.781,$$

$$\frac{r_{24}}{r_{12}} = \frac{r_{45}}{r_{12}} = 0.625, \qquad \frac{r_{25}}{r_{12}} = 1.249, \qquad \frac{r_{13}}{r_{12}} = 1.562;$$

and from the mass equations (12) we have,

$$\frac{m_1}{m_3} = 1, \quad \frac{m_2}{m_3} = \frac{m_5}{m_3} = \frac{m_5}{m_1} = 2.617, \quad \frac{m_4}{m_3} = 0.536, \quad \frac{m_5}{m_4} = 4.878, \quad \frac{m_2}{m_5} = 1.$$

As the point m_2 moves on towards p_4 and the point m_5 moves towards p_1, Fig. 7, the polygon moves through the square configuration (2), but this has already been discussed in Example 2 in connection with Fig. 5.

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